1·0

2·8   2·5

5·0 3·15   2·2

5·6
3·5

4·0   2·0

1·1

4·5

1·8

1·25   1·4   1·6

NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

MRC Technical Summary Report #1864

THE DYNAMICS OF PENSION FUNDING:
CONTRIBUTION THEORY

James C. Hickman, C. J. Nesbitt and
N. L. Bowers

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1978

(Received June 9, 1978)

Approved for public release
Distribution unlimited

78 09 12 035

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

THE DYNAMICS OF PENSION FUNDING: CONTRIBUTION THEORY

James C. Hickman, C. J. Nesbitt, N. L. Bowers

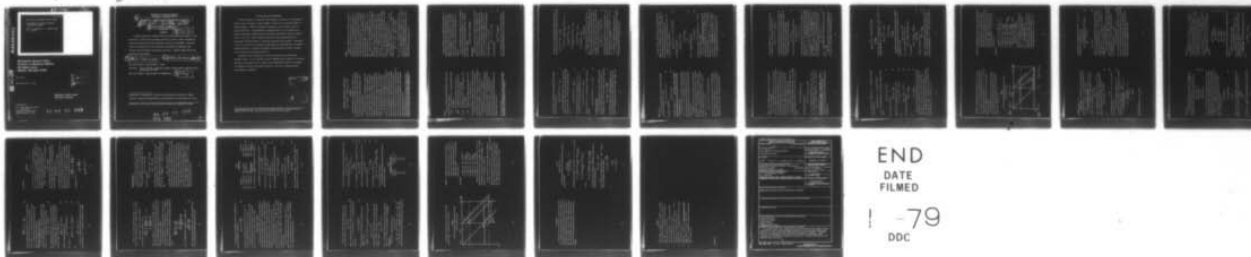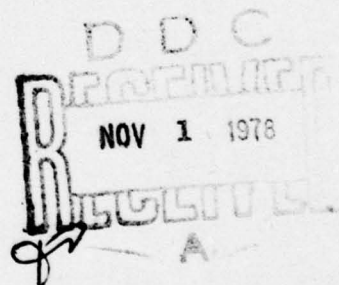Technical Summary Report #1864
July 1978

ABSTRACT

A general model for a pension plan involving growth with respect to the population, salaries and retirement benefits is used to study contribution patterns that may arise under different actuarial cost methods. Detailed results are presented for the case where the growth of population and salaries are described by exponential functions. Economic implications are presented and discussed.

(14) MRC-TSR-1864 / (15) DAAG 29-75-C-0024

(12) 22 p.

---

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48104.

[2] College of Business Administration, Drake University, Des Moines, IA 50311.

---

78 09 12 035
221 200

SIGNIFICANCE AND EXPLANATION

Pension funding, or actuarial cost, methods, are plans for accumulating funds to meet retirement income promises. There are an unlimited number of funding methods. However, because of regulation, only a few are used in business practice. The mathematical characteristics of an actuarial cost method include the ultimate level of required contributions and the ultimate level of the fund. These characteristics have been established for an environment characterized by a stationary population and economy. This paper is one of a series by the authors establishing the characteristics of pension funding methods in a dynamic environment characterized by growth in population, salaries and retirement benefits.

Perhaps the most interesting conclusion supports the conventional economic wisdom. If the interest rate on invested pension assets is less than the sum of the growth rates of salaries and populations, there is no longer an economic motivation for funding pensions. For this situation current cost funding is cheapest.

THE DYNAMICS OF PENSION FUNDING: CONTRIBUTION THEORY

James C. Hickman, C. J. Nesbitt[1] and N. L. Bowers[2]

I. INTRODUCTION

The paper "Introduction to the Dynamics of Pension Funding" [1] presented a mathematical model for a pure pension plan (no benefits other than for retirement) under conditions of growth in regard to the covered population and salaries. The model may be used to provide answers to a wide variety of pension funding questions.

In the earlier paper, the theory was developed for all members of the pension plan, both active and retired. In this paper the emphasis will shift to contribution theory and there is an advantage in developing the formulas taking into consideration only the subgroup of active members (see Richard K. Kischuk's discussion of [1]). The resulting formulas are somewhat simpler and provide answers to questions about contributions more readily than do those involving the whole group. In the final section of the paper, corresponding formulas for the whole group will be outlined.

In several developments in this paper exponential growth functions are used. This does not mean that it is anticipated that any real pension plan will have a covered population and corresponding salaries that will proceed smoothly along exponential paths. Instead, these special cases are developed because of their pedagogical value; they are easy to derive and interpret. In addition, the authors do not have the foresight to fix the jagged paths that the experience of real pension plans will probably follow. The long range cost estimates of the Social Security OASDI systems have been based on exponential economic assumptions for similar reasons.

The Model

A summary of the model, along the lines of that given in [1, Section II] is repeated here. All new entrants join the group at age a and all retirements occur at age r. Only retirement benefits are considered. For both active and retired participants, survivorship is deterministic and in accordance with the function ℓ_x which does not depend on the time variable t. At time zero, the density of new retirants at age r is ℓ_r and thereafter this density increases by a factor $g_1(t)$. This establishes a generation pattern of growth for participants. It is assumed that $g_1(t) > 0$. This implies a positive density $g_1(t + r - x)\ell_x$ of members age x, $a \leq x$, at time t. Salary rates at time zero are represented by the function $s(x)$, for a member age x, $a \leq x < r$. The function $s(x)$ captures the merit component of salary changes. Thereafter, salary rates increase by a factor $g_2(t)$. This establishes a year-of-experience pattern of growth of salaries. The function $g_2(t)$ is designed to capture the influence of productivity and inflation on salaries. The rate of initial annual pension payment, commencing at age r, is a fixed positive fraction b of the final salary rate. Pension payment rates increase during retirement by a factor $\beta(x)$.

We will now combine several of these functions. For $a \leq x < r$, the density of new pensions to be incurred at time $t + r - x$, in respect to the survivors of members age x at time t, is given by the function

$$h(t + r - x) = g_1(t + r - x)g_2(t + r - x)\ell_r s(r)b .$$

For $x \geq r$, $h(t + r - x)$ is the density of new pensions incurred at time $t - (x - r)$[1] for those who were then age r. Therefore, $h(t + r - x)(\ell_x/\ell_r)\beta(x)$ is the density of existing pensions for retirees age x at time t. In this expression, $\beta(x)$ is the aforementioned pension adjustment factor and $\beta(r) = 1$.

[1] In [1, p. 186] this time was incorrectly called t.

-2-

Outline

In Section II a number of pension funding functions in regard to active members are considered. These are:

(aā)(t), the present value at time t of future benefits for the then active members.[2]

p(t), the annual rate of normal cost for the plan at time t.

(aV)(t), the supplemental present value (accrued liability) of the plan at time t for the then active members.

(Pā)(t), the present value at time t of future normal costs for the then active members.

T_p(t), the annual rate of terminal funding normal cost for the plan at time t. This is a special case of p(t).

The basic income allocation equation, relative to active members is discussed.

In Section III ratios of the first four of the functions listed above to the fifth function are exhibited. These ratios take on particularly simple forms, that are easily interpreted, in the exponential growth case. That is, when the growth functions $g_1(t)$, $g_2(t)$ are exponential.

In Section IV various contribution rates determined by combinations of normal cost and rates of amortization of unfunded supplemental present value, or by aggregate funding are explored. In the exponential growth case, aggregate funding is found to achieve the same level of funding as some individual cost methods with particular amortization of the unfunded supplemental present value.

---

[2] In R. K. Kischuk's discussion of [1], the notation (Aā)(t) was used for this quantity. To avoid inconsistency with the notation (Pā)(t) and also annuity notation, the revised symbol (gā)(t) will be used.

Section V outlines some of the corresponding theory that emerges if the funding functions relate to the whole group of members, both active and retired, instead of to only the active group. In many ways this section provides a bridge to the developments in the original paper [1]. In this section it is natural to consider B(t), the annual rate of pension outgo at time t, in place of T_p(t), the annual rate of terminal funding normal cost, as a key function.

As in [1], the presentation will be mathematical. However, the mathematics is elementary and leads to natural verbal interpretations. It is hoped that other actuaries will become interested in developing further these ideas by means of numerical examples.

II. FUNDING FUNCTIONS FOR ACTIVE MEMBERS

Basic Functions

Central to the study of contribution theory is the annual rate of terminal funding cost for the plan at time t. This rate will be denoted by $T_p(t)$. It will serve as a building block and as a standard of comparison for other contribution patterns. For the model plan, the density of pensions expected to arise $(r - x)$ years later with respect to members age x at time t, $a \leq x \leq r$ is $h(t + r - x)$. Therefore, the annual rate of terminal funding cost for the plan at time $t + r - x$ will be

$$T_{p(t + r - x)} = h(t + r - x)\bar{a}_x^\beta ,$$ (1)

where the annuity symbol

$$\bar{a}_x^\beta = \int_r^\infty e^{-\delta(x-r)} {}_{x-r}P_r\, \beta(x)dx ,$$ (2)

and δ is the annual force of interest. Note that the annuity function incorporates the pension adjustment function $\beta(x)$. With the terminal funding

cost rate function, it is now easy to define the present value of future benefits for the active group as

$$(a\bar{A})(t) = \int_a^r e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)dx \ . \tag{3}$$

By using the pension purchase density function m(x) and the accrual function M(x) (see [1, p. 182] and [3]), we may write immediately the annual normal cost rate at time t for the actuarial cost method associated with m(x) as

$$P(t) = \int_a^r e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)m(x)dx \ . \tag{4}$$

Then the supplemental present value (accrued liability) at time t for active members, for the actuarial cost method described by m(x), is

$$(a\bar{V})(t) = \int_a^r e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)M(x)dx \ . \tag{5}$$

The present value at time t of future normal costs of the plan for active members is

$$(\underline{Pa})(t) = \int_a^r e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)(1 - M(x))dx \ . \tag{6}$$

Income Allocation Equation

In [1] a liability growth equation in regard to all members, both active and retired, was expressed in formula (40) and rearranged in formula (43). Corresponding to formula (43), we have the equation, relevant to active members,

$$P(t) + \delta(a\bar{V})(t) = T_{\underline{p}}(t) + \frac{d(a\bar{V})(t)}{dt} \ . \tag{7}$$

This equation may be described as an income allocation equation: the normal cost and assumed interest are allocated to the terminal funding cost and the change in the supplemental present value for active members. Equation (7) is

very general. Using the notation developed to describe our basic model, we may verify formula (7) by using integration by parts on formula (4) as follows:

$$P(t) = \int_a^r e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)m(x)dx$$

$$= -e^{-\delta(r-x)} T_{\underline{p}}(t+r-x)M(x)\Big|_a^r - \int_a^r M(x)e^{-\delta(r-x)}d\, T_{\underline{p}}(t + r - x)$$

$$- \int_a^r M(x)\delta e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)dx$$

$$= T_{\underline{p}}(t) + \frac{d}{dt}\int_a^r M(x)e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)dx$$

$$- \delta \int_a^r M(x)e^{-\delta(r-x)} T_{\underline{p}}(t + r - x)dx$$

$$= T_{\underline{p}}(t) + \frac{d}{dt}(a\bar{V})(t) - \delta(a\bar{V})(t) \ .$$

III. FUNCTION RATIOS

In this section ratios of the functions $(a\bar{A})(t)$, $P(t)$, $(a\bar{V})(t)$ and $(\underline{Pa})(t)$ to $T_{\underline{p}}(t)$, the annual rate of terminal funding are considered. These ratios lead to insights because, according to the income allocation equation (7), $T_{\underline{p}}(t)$ may be thought of as the outgo function in regard to funding for active members. First, we have the ratio of the value of future benefits for the active group to the annual rate of terminal funding cost

$$(a\bar{A})(t)/T_{\underline{p}}(t) = \int_a^r [h(t + r - x)/h(t)]e^{-\delta(r-x)}dx \ , \tag{8}$$

which may be viewed as the present value of a varying annuity certain. In the exponential growth case where $g_1(t) = e^{\alpha t}$ (population growth), $g_2(t) = e^{\gamma t}$ (salary growth) and $h(t) = g_1(t)g_2(t)\ell_s(r)b = e^{\tau t}\ell_s(r)b$, $\tau = \alpha + \gamma$, then $[h(t + r - x)/h(t)]e^{-\delta(r-x)} = e^{-(\delta-\tau)(r-x)}$.

At this stage it is necessary to distinguish the three cases $\delta > \tau$, $\delta = \tau$ and $\delta < \tau$. In the next three subsections these three cases will be explored and the analyses extended to the other ratios of interest.

Case 1. $\delta > r$

In this case the annual force of interest is greater than the combined rates of salary and population growth. We have

$$[h(t + r - x)/h(t)]e^{-\delta(r-x)} = v^{r-x}, \qquad (9)$$

and formula (8) simplifies to

$$(a\underline{A})(t)/{}^T\underline{P}(t) = \bar{a}_{\overline{r-a}} \qquad (10)$$

evaluated at annual force $\theta = \delta - r$. Here, and in the sequel, if the force to be used in the evaluation of a compound interest function is not stated, it is to be taken as $\theta = \delta - r$.

It is clear that in this case $(a\underline{A})(t)/{}^T\underline{P}(t)$ is a decreasing function of $\theta = \delta - r$. If $r < 0$, that is if the combination of population and salaries is on a decreasing rather than increasing exponential path, the ratio of the present value of future benefits to the current terminal cost normal cost rate may be fairly small.

From formula (10) one sees that the present value of future benefits for active members is equal to the discounted value at the force θ of an annuity of ${}^T\underline{P}(t)$ for $r - a$ years. More completely, in the exponential growth case $(a\underline{A})(t)$ equals the discounted value at annual force δ of an increasing annuity with payment rate from formula (1) of ${}^T\underline{P}(t + u) = {}^T\underline{P}(t)e^{ru}$

at time $t + u$, $0 \le u < r - a$. That is

$$(a\underline{A})(t) = \int_0^{r-a} {}^T\underline{P}(t + u)e^{-\delta u}du = \int_0^{r-a} ({}^T\underline{P}(t)e^{ru})e^{-\delta u}du .$$

The key point is that in the exponential growth case, the terminal funding normal cost increases at the total growth rate $r = \alpha + \gamma$, the sum of the population and salary growth rates. Note also that the ratio in formula (10) is independent of t as a result of the exponential growth functions.

The function $(a\underline{A})(t)$, the present value of future benefits for active members at time t, is independent of the actuarial cost method. However, the remaining funding functions defined by formulas (4), (5) and (6) depend on the actuarial cost method through the accrual function $M(x)$.

For example, the ratio of the annual normal cost rate at time t to the annual rate of terminal cost funding is

$$\underline{P}(t)/{}^T\underline{P}(t) = \int_a^r [h(t + r - x)/h(t)]e^{-\delta(r-x)} M(x)dx . \qquad (11)$$

In the exponential growth case

$$\underline{P}(t)/{}^T\underline{P}(t) = \int_a^r v^{r-x} M(x)dx$$

$$= v^r \int_a^r e^{x(\delta-r)} M(x)dx$$

$$= v^{r-\hat{x}} . \qquad (12)$$

Here $\hat{x}$ is calculated from the equation

$$e^{\delta \hat{x}} = \int_a^r e^{\delta x} M(x)dx . \qquad (13)$$

The existence of a value of $\hat{x}$ on the interval from a to r is assured by the mean value theorem for integrals.

We will call $\hat{x}$ the _average age of normal cost payment_ associated with the actuarial cost method defined by $M(x)$ and the combination of interest, population, and salary forces $\theta = \delta - r = \delta - \alpha - \gamma$. Two extreme cases need special attention. For terminal funding $M(x) = 0$, $a \le x < r$, and $M(x) = 1$. $r \le x$, and $\hat{x} = r$. For _initial funding_, the whole pension cost is funded for each entrant by a lump sum payment at entry, $M(x) = 1$, $a \le x$, and $\hat{x} = a$. In the setting of the model plan with exponential growth, the

number $\hat{x}$ can tell us some of the characteristics of the actuarial cost method with which it is associated.[3]

Equation (13) has an interesting interpretation. Note that the right hand side, to be denoted by $\psi(\theta)$, may be interpreted as the moment generating function associated with the actuarial cost method defined by $m(x)$. This leads to the formal conclusion that if two actuarial cost methods yield the same value of $\hat{x}(\theta)$, for all values of θ on an interval containing zero, then their associated moment generating functions are the same on the interval, and the two actuarial cost methods are identical.

It is of more practical use to examine the relationship between $\hat{x}$ and the characteristics of the associated actuarial cost method. For two continuous pension purchase density functions, $m(x)$ and $m_1(x)$ with $M(a) = M_1(a) = 0$ we have

$$e^{\theta\hat{x}} - e^{\theta\hat{x}_1} = \int_a^r e^{\theta x} d[M(x) - M_1(x)] .$$

Using integration by parts we have

$$e^{\theta\hat{x}} - e^{\theta\hat{x}_1} = -\theta \int_a^r [M(x) - M_1(x)] e^{\theta x} dx .$$

If $m(x)$ is associated with a decelerating cost method ($m'(x) < 0$), and $m_1(x)$ with an accelerating cost method ($m_1'(x) > 0$), then $M(x) - M_1(x) > 0$, $a < x < r$ and for θ positive or negative we have $\hat{x} < \hat{x}_1$. We are led to

[3] For a given value of θ, the number $\hat{x}$, which depends on θ, does not characterize an actuarial cost method. For fixed θ and $\hat{x}$, $a < \hat{x} < r$, it is easy to construct a density function $m(x)$ that will yield $\int_a^r e^{\theta x} m(x) dx = e^{\theta\hat{x}}$ by using step functions or exponential density functions as described in [3].

conclude the following obvious result: In comparing decelerating $(m(x))$ and accelerating $(m_1(x))$ cost methods we have from formula (12)

$$P(t)/{}^T P(t) = v^{r-\hat{x}} < v^{r-\hat{x}_1} = P_1(t)/{}^T P_1(t) ,$$

or that the ratio of annual normal cost rate to the terminal cost funding rate is less for decelerating cost methods than for accelerating cost methods. A closely related argument will be used in the next subsection in comparing the prorata accrued benefit and entry age normal cost methods.

Using our new symbol, formula (12) may be rearranged as

$$P(t) = v^{r-\hat{x}} {}^T P(t) , \tag{14}$$

or by the use of formula (1) for the exponential growth case as

$$P(t) = e^{-\delta(r-\hat{x})} e^{r(r-\hat{x})} {}^T P(t)$$
$$= e^{-\delta(r-\hat{x})} {}^T P(t+r-\hat{x}) . \tag{15}$$

This shows that the annual normal cost rate at time t may be thought of as remaining in the fund for $(r-\hat{x})$ years and then utilized to provide terminal funding cost ${}^T P(t+r-\hat{x})$. The total funding term for current active members is $(r-a)$ years. We shall see that this may be thought of as consisting of a past funding term of $(r-\hat{x})$ years and a future funding term of $N = \hat{x} - a$ years. Of course, $\hat{x}$ depends on the actuarial cost method. In the special case of terminal funding $\hat{x} = r$, that is the past funding term for the active group is zero years and all funding is in the future. For initial funding

$$I_P(t) = e^{-\delta(r-a)} {}^T P(t+r-a) , \tag{16}$$

and $\hat{x} = a$. Thus the future funding term is zero years and the past funding term $r - a$ years.

The next ratio we will consider is the ratio of the supplemental present value at time t for active members to the terminal funding cost rate at

time t. We remain in the situation where $\delta > \tau$. We have for a cost method characterized by $M(x)$.

$$(aV)(t)/{}^Tp(t) = \int_a^r [h(t + r - x)/h(t)]e^{-\delta(r-x)} M(x)dx .$$

This ratio may be calculated directly in the exponential growth case from the equation

$$(aV)(t)/{}^Tp(t) = \int_a^r v^{r-x} M(x)dx . \qquad (17)$$

but it is simpler to note that

$$\frac{d}{dt}(aV)(t) = \tau(aV)(t) . \qquad (18)$$

When this result is substituted into equation (7), we have

$$p(t) + \delta(aV)(t) = {}^Tp(t) + \tau(aV)(t) , \qquad (19)$$

or

$$(aV)(t) = [{}^Tp(t) - p(t)]/(\delta - \tau) . \qquad (20)$$

By use of formula (14) this becomes

$$(aV)(t) = {}^Tp(t)\bar{a}_{\overline{r-\hat{x}}|} \qquad (21)$$

or

$$(aV)(t) = p(t)\bar{s}_{\overline{r-x}|} . \qquad (22)$$

Formula (21) exhibits the supplemental present value (accrued liability) for active members as the present value of the current terminal funding cost over the next $(r - \hat{x})$ years. Formula (22) looks at the same quantity as the accumulated value of current normal costs for the past $(r - \hat{x})$ years. These interpretations of formulas (21) and (22) are somewhat incomplete. Stated another way, $(aV)(t)$ is the present value at force of interest δ of the terminal funding cost ${}^Tp(t + u) = {}^Tp(t)e^{\tau u}$, $0 \leq u \leq r - \hat{x}$.

$$(aV)(t) = \int_0^{r-\hat{x}} {}^Tp(t + u)e^{-\delta u}du = \int_0^{r-\hat{x}} {}^Tp(t)e^{\tau u}e^{-\delta u}du$$

$$= {}^Tp(t)\bar{a}_{\overline{r-\hat{x}}|} .$$

We also have that $(aV)(t)$ is the accumulated value, at force of interest δ, of the annual normal costs $p(t - u) = p(t)e^{-\tau u}$, $0 \leq u \leq r - \hat{x}$,

$$(aV)(t) = \int_0^{r-\hat{x}} p(t - u)e^{\delta u}du$$

$$= \int_0^{r-\hat{x}} p(t)e^{-\tau u}e^{\delta u}du$$

$$= p(t)\bar{s}_{\overline{r-\hat{x}}|} .$$

The present value of benefits for active members must equal the supplemental present value plus the present value of future normal costs. That is adding formulas (5) and (6),

$$(aA)(t) = (aV)(t) + (pa)(t) .$$

One may now apply formulas (10) and (21) to show

$$(aA)(t) = {}^Tp(t)\bar{a}_{\overline{r-a}|} - {}^Tp(t)\bar{a}_{\overline{r-\hat{x}}|} + (pa)(t) .$$

and by rearranging

$$(pa)(t) = {}^Tp(t)\bar{a}_{\overline{r-a}|} - {}^Tp(t)\bar{a}_{\overline{r-\hat{x}}|}$$

$$= {}^Tp(t)v^{r-a}\bar{s}_{\overline{\hat{x}-a}|} . \qquad (23)$$

An application of formula (14) yields

$$(pa)(t) = p(t)\bar{a}_{\overline{\hat{x}-a}|} . \qquad (24)$$

Formula (23) exhibits the present value of future normal costs for active lives at time t as the present value of terminal funding costs that will arise during the future funding term of $N = \hat{x} - a$ years. Formula (24) expresses the
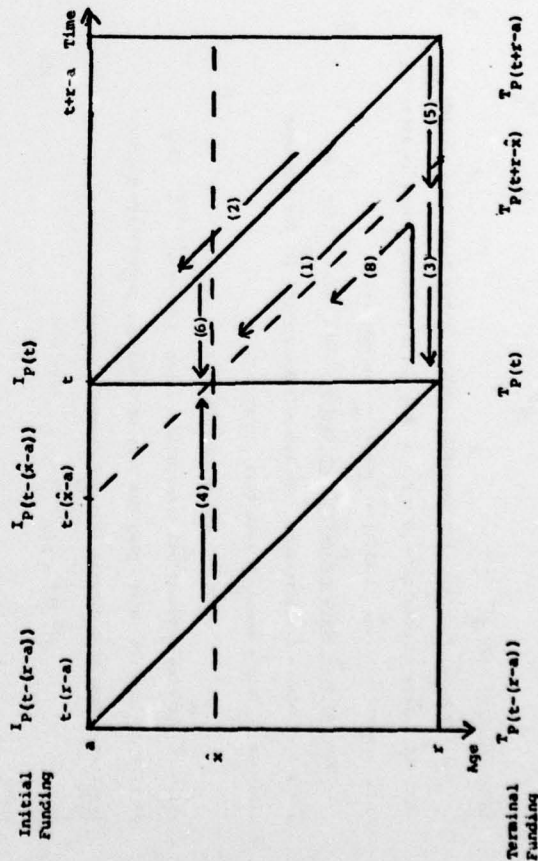
same quantity as the present value of normal costs at rate $P(t)$ for the next $N = \hat{x} - a$ years. This value will provide the terminal funding costs $(r - \hat{x})$ years later. In reviewing these interpretations, recall that the compound interest functions are evaluated at $\delta - \tau = \delta - a - \gamma$, that is each interest function depends on the annual forces of interest, population change, and salary change.

These concepts may be clarified by the Lexis type diagram in Figure 1. Cohorts of entrants may be viewed as moving along diagonal lines in the figure. Vertical lines depict cross sectional views of the funding for active lives at a fixed point of time.

Figure 1

Illustration of Relations Among Funding Functions, Active
Lives, Exponential Growth

Initial Funding $\quad$ $I_{P(t-(r-a))}$ $\quad$ $I_{P(t-(\hat{x}-a))}$ $\quad$ $I_{P(t)}$

Terminal Funding $\quad$ $T_{P(t-(r-a))}$ $\quad\quad$ $T_{P(t)}$ $\quad\quad$ $T_{P(t+r-\hat{x})}$ $\quad$ $T_{P(t+r-a)}$

Explanation: The dashed diagonal line may be viewed as the mean path followed by the group active at time t. The paths to be followed in tracing the relationships are indicated by arrows indexed by relationship numbers. Along diagonal lines the final salary and population growth functions are fixed, only interest contributes to changed values. Payments along horizontal lines are of different amounts because of τ and have different present values because of δ.

Relationships:

(1) Equation (15), $\quad P(t) = e^{-\delta(r-\hat{x})} T_P(t + r - \hat{x})$

(2) Equation (16), $\quad I_P(t) = e^{-\delta(r-a)} T_P(t + r - a)$

(3) Equation (21), $\quad (aV)(t) = T_P(t)\bar{a}_{\overline{r-\hat{x}}}$

(4) Equation (22), $\quad (aV)(t) = P(t)\bar{a}_{\overline{r-\hat{x}}}$

(5) Equation (23), $\quad (Pa)(t) = T_P(t)e^{-(\delta-\tau)(r-\hat{x})}\bar{a}_{\overline{r-a}}$

(6) Equation (24), $\quad (Pa)(t) = P(t)\bar{a}_{\overline{\hat{x}-a}}$

(7) Equation (10), $\quad (aA)(t) = T_P(t)\bar{a}_{\overline{r-a}}$, same as relationships (3) and (5) combined.

(8) Equation (12), $\quad P(t) = e^{-(\delta-\tau)(r-\hat{x})} T_P(t)$

Because of the interpretations provided by relationships (4) and (6), $r - \hat{x}$ was called the past funding term and $N = \hat{x} - a$ the future funding term. Since $(aA)(t) = (aV)(t) + (Pa)(t)$, we have a natural division of the present value of future benefits for active lives into a past and future component and of the funding term $(r - a)$ into associated terms of length $(r - \hat{x})$ and $N = \hat{x} - a$.

Application of Case 1. ($\delta > \tau$)

a). Terminal Funding. For terminal funding, it has been noted already that the future funding term $N = r - a$ and the past funding term is zero.

Then $(aV)(t) = 0$ and $(Pa)(t) = {}_{T}P(t)\bar{a}_{\overline{r-a}|} = (aA)(t)$.

b). Pro Rata Accrued Benefit Cost Method. Here $m(x) = 1/(r - a)$, $a < x < r$, $m(x) = 0$, elsewhere. By Jensen's inequality [4] for a random variable x and a function such that $\psi''(x) > 0$,

$$E[\psi(X)] > \psi(E[X]) . \tag{25}$$

where E denotes mathematical expectation. Take $\psi(x) = e^{(\delta-\tau)x}$, then $\psi''(x) > 0$, and let $m(x)$ play the role of a uniform probability density function. Then from formula (13) and Jensen's inequality

$$e^{(\delta-\tau)\hat{x}} = E[e^{(\delta-\tau)X}] > e^{(\delta-\tau)E(X)} \tag{26}$$

and

$$\hat{x} > E[X] .$$

With the uniform density $m(x) = 1/(r - a)$, $\hat{x}[X] = (a + r)/2$. We may then rewrite inequality (26) as

$$\hat{x} \text{ (for pro rata accrued benefit cost method)} > (a + r)/2 . \tag{27}$$

In other words, for this cost method, the future funding term N is more than one-half of the total funding term of $r - a$ years.[4]

c). Entry Age Normal Cost Method. A choice of $m(x)$ is

$$m(x) = \frac{e^{-(\delta-\gamma)x}x\,l_x\,s(x)}{\int_a^r e^{-(\delta-\gamma)y}y\,l_y\,s(y)dy} , \quad a \leq x < r$$

$$= 0, \quad \text{elsewhere} .$$

---

[4] This proof was suggested by Dr. Hans U. Gerber.

Then from formula (4), after substitution for ${}_{T}P(t + r - x)$ and $m(x)$ in the exponential case and rearranging, we have

$$P(t) = \left[\frac{e^{-(\delta-\gamma)r}s(r)l_r\,b_r\,\beta}{\int_x^r e^{rt+\alpha(r-x)}l_x\,s(x)dx}\right]\Big/ \int_r e^{-(\delta-\gamma)y}s(y)l_y\,dy \tag{28}$$

or

$$P(t) = S_{r_a} W(t) . \tag{29}$$

In formula (29) S_{r_a} is the entry age normal cost rate, as a level fraction of salary, for an employee entering at age a, retiring at age r, and having an annual salary rate $e^{\gamma(x-a)}s(x)$ at age x, $a \leq x < r$. The symbol $W(t)$ is the annual payroll at time t (see [1, formula (77)]). Note that S_{r_a} is independent of t and of the population growth rate α.

Here a direct application of Jensen's inequality does not lead to a statement about $\hat{x}$, the average age of normal cost payment. However, an indirect application (see Appendix) shows that, if $e^{-\alpha x}s(x)l_x$ is a decreasing function of x, and, as throughout this subsection, $\delta > \tau$, then

$$\hat{x} \text{ (entry age normal method)} < (r + a)/2 . \tag{30}$$

That is, under the stated condition, for entry age normal funding the future funding period $N = \hat{x} - a$ is less than one-half the entire funding term $(r - a)$. It follows from formulas (14), (27) and (30) that the normal cost for the entry age normal method is less than for the accrued benefit method, and from formula (21) the reverse relation holds for the supplemental present values.

Case 2, $\delta = \tau$.

In this case the annual force of interest equals the combined rate of salary and population growth. We have $[h(t + r - x)/h(t)]e^{-\delta(r-x)} = 1$ and formulas (10), (21), (12) and (23) may be respectively replaced by

$$(_a\ddot{A})(t) = {}^T\underline{p}(t)(r - a) \tag{31}$$

$$(_a\ddot{V})(t) = {}^T\underline{p}(t)(r - \bar{x}) \tag{32}$$

$$\underline{P}(t) = {}^T\underline{p}(t) \tag{33}$$

$$(\underline{Pa})(t) = {}^T\underline{p}(t)(\bar{x} - a) \tag{34}$$

were $r - \bar{x} = r - \int_a^r x m(x)dx = \int_a^r M(x)dx$. These formulas relate to certain of the formulas in [1, Section VI, The Exponential Growth Case] in the following fashion: Formulas (31), (32) and (33) are the active members analogues of formulas (89), (90) and (88) in [1]; formula (34) is identical to (92) of [1].

When $\theta = \delta - \tau = 0$ formula (13) does not lead to a definition of $\hat{x}$.

There is a natural way out of this difficulty that leads to insights about $\hat{x}$. From formula (13) we have

$$e^{\theta\hat{x}} = \int_a^r e^{\theta x} m(x)dx$$

$$\hat{x} = \frac{1}{\theta} \ln\left[\int_a^r e^{\theta x} m(x)dx\right] \tag{35}$$

$$\hat{x} = C(\theta)/\theta ,$$

where $C(\theta) = \ln\psi(\theta)$ is the cumulant generating function associated with the density $m(x)$ [2, p. 307]. Now we define $\hat{x}$ in the case $\theta = 0$ as

$$\hat{x} = \lim_{\theta \to 0} C(\theta)/\theta$$
$$= \bar{x} .$$

The evaluation of $\lim_{\theta \to 0} C(\theta)/\theta$ requires one application of L' Hospital's rule.

Case 3, $\delta < \tau$.

a) **Basic ratios.** This is the case where $\tau = \alpha + \gamma$; the sum of the generation growth rate α and γ the salary growth rate, is such that $\delta < \tau$. In this case the basic mathematical formulas remain. The number $\hat{x}$ is still defined by formula (13) for each actuarial cost method. However, the formulas involving compound interest functions take on a new significance for $v = e^{-(\delta-\tau)} = e^{-(\tau-\delta)} > 1$ and where we had $\ddot{a}_{\overline{n}|}$ evaluated at force $\delta - \tau > 0$, we now have $\bar{s}_{\overline{n}|}$ evaluated at $\tau - \delta > 0$. In general where a discount effect was observed in Case 1, an accumulation effect is observed in Case 3.

Corresponding to formulas (10), (14), (21), (22) and (24) respectively we now have

$$(_a\ddot{A})(t) = {}^T\underline{p}(t)\bar{s}_{\overline{r-a}|} \tag{36}$$

$$\underline{P}(t) = {}^T\underline{p}(t)e^{(\tau-\delta)(r-\hat{x})} \tag{37}$$

$$(_a\ddot{V})(t) = {}^T\underline{p}(t)\bar{s}_{\overline{r-\hat{x}}|} \tag{38}$$

$$(_a\ddot{V})(t) = \underline{p}(t)\ddot{a}_{\overline{r-\hat{x}}|} \tag{39}$$

$$(\underline{Pa})(t) = \underline{p}(t)\bar{s}_{\overline{N}|} \tag{40}$$

where each of the compound interest functions are evaluated at annual force of interest $\tau - \delta$.

The application of Jensen's inequality yields

$\hat{x}$ (for pro rata accrued benefit cost method) $< (a + r)/2$.

This is the opposite of the relation for $\delta > \tau$ found in inequality (27). It seems difficult to obtain a useful general statement about $\hat{x}$ for the entry age normal cost method in this case.

b) **Income allocation.** The income allocation equation (19) takes on the form

$$\underline{P}(t) = {}^T\underline{p}(t) + (\tau - \delta)(_a\ddot{V})(t) .$$

That is, the normal cost must provide not only the terminal funding cost, but also for the growth required in the supplemental present value, $(aV)(t)$, in excess of interest income. In Case 3, terminal funding has the lowest cost among the methods that complete funding by age r. Further, if one cost method defines a higher supplemental value than a second cost method, the normal cost for the first method will have a _higher normal cost than the_ second. Thus initial funding would in Case 3 result in not only the highest supplement present value $(aV)(t)$ but also the highest normal cost, among the cost methods completing funding during the working lifetime of members.

c) _Discussion._ The practical implications of interest rates below the sum of the growth rates of salaries and population are enormous. Traditional economic arguments in favor of funded pensions begin to lose their validity. The change to current cost funding of pensions _in nations with a high rate of wage inflation and relatively low interest rates_ would seem to be a realization of this theoretical result. (See [7] for a discussion of this important point.)

In this paper the stress has been on contribution theory. Current cost, or pay-as-you-go funding, has not been one of the cost methods under considera- tion. We have not made any assumptions about the benefit adjustment function $\delta(x)$. However, it is clear that there are many patterns of post-retirement benefit adjustments that would make the current cost rate below the terminal funding cost rate.

IV. CONTRIBUTION THEORY

In this section several patterns of contribution rates are developed. These patterns are selected to build up funds to meet the cost of the plan in regard to active members. With slight changes, the theory could be developed for the whole group, active or retired. The contribution patterns will be related to actuarial cost methods of the individual type. While the patterns could be developed in a more general context, the exponential growth case

-19-

will be assumed throughout. Amortization factors $(1/\bar{a}_{\overline{n}})$ will be evaluated at force $\theta = \delta - \tau$ to provide for amortization of unfunded supplemental present value as a level percentage of payroll which is increasing at an annual rate $\tau = \alpha + \gamma$, the sum of the population and salary growth rates. This will be done in lieu of considering amortization by level amounts.

Normal Cost Plus Amortization Over Fixed Term

The objective is to reach fully funded status for some actuarial cost method at the end of n years measured from an arbitrary initial time. For convenience the initial time will be denoted by zero. The annual contribution rate $(aC)(t)$ at time t, $0 \leq t < n$, in regard to active members is denoted by the formula

$$(aC)(t) = P(t) + .01f \cdot W(t) .$$

In this equation $W(t)$ is the rate of payroll payment at time t (see formula (77) in [11]), and f is a level percentage of payroll determined so as to amortize the initial unfunded supplemental present value over n years.

If $(aF)(t)$ denotes the fund available for the active members at time t, and $(aU)(t) = (aV)(t) - (aF)(t)$ is the _unfunded supplemental present value_ for active members at time t, then

$$(aU)(0) = (aV)(0) - (aF)(0) = .01f \int_0^n e^{-\delta t} W(t)dt$$

$$= .01f W(0) \int_0^n e^{-\delta t} e^{\tau t}dt$$

$$= .01f W(0)\bar{a}_{\overline{n}} .$$

where $\bar{a}_{\overline{n}}$ is calculated at force $\delta - \tau$ as previously noted. In this result we have used the fact that $W(t) = e^{\tau t}W(0)$, and in developing equation (4U) we shall also use the fact that $P(t) = e^{\tau t}P(0)$. These results are

-20-

achieved in [1, page 199]. Now substituting into our expression for the contribution rate, we have

$$(_a\ddot{C})(t) = \underline{P}(t) + \frac{(_a\ddot{U})(0)}{_a\overline{n|}} \cdot \frac{W(t)}{W(0)}$$

$$= \underline{P}(t) + \frac{(_a\ddot{U})(0)}{_a\overline{n|}} \cdot e^{rt}$$

$$= \left[\underline{P}(0) + \frac{(_a\ddot{U})(0)}{_a\overline{n|}}\right] e^{rt}. \quad (41)$$

From these formulas it is clear that $(_a\ddot{C})(t)/W(t)$ is a constant, namely

$$\left[\underline{P}(0) + \frac{(_a\ddot{U})(0)}{_a\overline{n|}}\right] W(0)^{-1}.$$

We turn now to the question of how convergence to a fully funded status occurs. With contributions determined by $(_a\ddot{C})(t)$ as in formula (41), the active member fund $(_a\underline{F})(t)$ grows according to the differential equation

$$\frac{d(_a\underline{F})(t)}{dt} = (_a\ddot{C})(t) + \delta(_a\underline{F})(t) - T_{\underline{P}}(t)$$

$$= \underline{P}(t) + \frac{(_a\ddot{U})(0)}{_a\overline{n|}} e^{rt} + \delta(_a\underline{F})(t) - T_{\underline{P}}(t). \quad (42)$$

The income allocation equation (7) may be rearranged as

$$\frac{d(_a\underline{V})(t)}{dt} = \underline{P}(t) + \delta(_a\underline{V})(t) - T_{\underline{P}}(t).$$

When equation (42) is subtracted from this expression, the result is

$$\frac{d(_a\ddot{U})(t)}{dt} = \delta(_a\ddot{U})(t) - \frac{(_a\ddot{U})(0)}{_a\overline{n|}} e^{rt}. \quad (43)$$

Changing t to h, multiplying through by the integrating factor $e^{-\delta h}$, and rearranging yields

$$d[e^{-\delta h}(_a\ddot{U})(h)] = -\frac{(_a\ddot{U})(0)}{_a\overline{n|}} e^{-(\delta-r)h} dh.$$

Integrating from 0 to t and solving for $(_a\ddot{U})(h)$ yields

$$(_a\ddot{U})(t) = (_a\ddot{U})(0)e^{\delta t} - (_a\ddot{U})(0)e^{rt} \overline{s}_{t|} \overline{s}_{n|}^{-1}$$

$$= (_a\ddot{U})(0)e^{rt}[e^{(\delta-r)t} - \overline{s}_{t|} \overline{s}_{n|}^{-1}]$$

$$= (_a\ddot{U})(0)e^{rt} \overline{a}_{n-t|} / \overline{a}_{n|}. \quad (44)$$

It may be verified that equation (44) holds whether $\delta >$, = or $< r$. It is now clear from equation (44) that $(_a\ddot{U})(n) = 0$, which implies that $(_a\underline{F})(n) = (_a\underline{V})(n)$. Also from equations (43) and (44) we note that

$$\frac{d(_a\ddot{U})(t)}{dt} = (_a\ddot{U})(t)[\delta - \overline{a}_{n-t|}^{-1}], \quad (45)$$

and if $\overline{s}_{n|}^{-1} < \delta$, then $d(_a\ddot{U})(t)/dt > 0$ until time t_0, when $\overline{a}_{n-t_0|}^{-1} = \delta$. That is $(_a\ddot{U})(t)$ increases to a maximum which occurs at time t_0 and then decreases.

In practice one might select what appears to be a more flexible contribution system, to be denoted by super-imposing tildes on the functions, such that

$$(_a\tilde{C})(t) = \underline{P}(t) + (_a\tilde{U})(t)/\overline{a}_{n-t|}, \quad (46)$$

with $(_a\tilde{U})(0) = (_a\underline{V})(0) - (_a\underline{F})(0) = (_a\ddot{U})(0)$. Again the objective is to amortize the unfunded supplemental present value over a fixed n years. In applications equation (46) allows for spreading experience gains and losses. However, in the deterministic model we are studying, gains and losses do not appear, and we might guess that identical results will be produced by contributions defined by equations (41) and (46). To confirm this guess we could write the equation analogous to equation (42), subtract it from the income allocation equation and obtain

$$\frac{d(a\ddot{U})(t)}{dt} = -(a\ddot{U})(t)(\delta - \bar{a}_{\overline{n-t}|}^{-1}),$$ (47)

which is the same differential equation as (45). Since the two differential equations involve the same initial condition, we can conclude that $(a\ddot{U})(t) = (a\ddot{U})(t)$, $0 \le t \le n$. This in term implies that $(a\ddot{C})(t) = (a\ddot{C})(t)$, $0 \le t \le n$, as may be confirmed by substitution from equation (44) into equation (46) and comparing with equation (41).

Normal Cost Plus Amortization Over a Moving Term

In this case, the objective is to attain fully funded status for some actuarial cost method asymptotically. The annual contribution rate $(a\underline{C})(t)$ at time t, $0 \le t$, for active members is defined by the formula

$$(a\underline{C})(t) = P(t) + (a\underline{U})(t)/\bar{a}_{\overline{n}|}.$$ (48)

That is, amortization is over a term of n years from the current time t rather than from time zero. The amortization term continually moves forward.

The formula analogous to formula (43), derived by a similar chain of steps, using $(a\underline{C})(t)$ as in equation (48), is

$$\frac{d}{dt}(a\underline{U})(t) = -(\bar{a}_{\overline{n}|}^{-1} - \delta)(a\underline{U})(t).$$ (49)

Solving this differential equation yields

$$(a\underline{U})(t) = (a\underline{U})(0)\exp(-(\bar{a}_{\overline{n}|}^{-1} - \delta)t).$$ (50)

Now if $\bar{a}_{\overline{n}|}^{-1} > \delta$,

$$\lim_{t\to\infty}[(a\underline{V})(t) - (a\underline{F})(t)] = \lim_{t\to\infty}(a\underline{U})(t) = 0$$ (51)

and

$$(a\underline{F})(t) \to (a\underline{V})(t) \text{ as } t \to \infty.$$ (52)

It is clear that convergence, as described in equation (52) will occur for certain sets of (n,δ,τ). For example, the conditions for convergence

may be written

$$n < \underbrace{\begin{cases} \dfrac{\ln \delta - \ln \tau}{\delta - \tau}, & \delta \ne \tau \\[2ex] \dfrac{1}{\delta}, & \delta = \tau \end{cases}}$$

If $\delta > 0$ and $\tau = 0$, any n, $0 < n < \infty$ will assure convergence. If $\delta = \tau$, $n < 1/\delta$, and as $\tau \to \infty$, the upper bound on n approaches zero. The point is that if pension obligations are growing very rapidly as a result of a large total growth rate relative to the interest rate, to assure convergence n, the rolling amortization period, must be small.

If $\bar{a}_{\overline{n}|}^{-1} = \delta$ the functions $(a\underline{F})(t)$ and $(a\underline{V})(t)$ are unbounded, but

$$(a\underline{U})(t) = (a\underline{U})(0),$$ (53)

and no progress is made toward reducing the unfunded present supplemental value in an absolute sense. However, in a relative sense progress is made. This may be seen by rearranging formula (53) as

$$(a\underline{V})(t) - (a\underline{F})(t) = (a\underline{V})(0) - (a\underline{F})(0).$$

In the exponential growth case, we may observe that equation (21) tell us that $(a\underline{V})(t)$ grows at the same rate as terminal funding cost $T_P(t)$. From equation (6) we learn that $T_P(t)$ grows as $h(t)$ and in the exponential growth case $h'(t) = \tau h(t)$. Therefore, $(a\underline{V})(t) = (a\underline{V})(0)e^{\tau t}$ and we may divide each side of our reformed equation (53) by this expression to obtain

$$\left[1 - \frac{(a\underline{F})(t)}{(a\underline{V})(t)}\right] = \left[1 - \frac{(a\underline{F})(0)}{(a\underline{V})(0)}\right]e^{-\tau t}.$$

This implies that if $\tau > 0$

$$\lim_{t\to\infty}\left[1 - \frac{(a\underline{F})(t)}{(a\underline{V})(t)}\right] = 0$$

or

$$\lim_{t\to\infty}[(a\underline{F})(t)][(a\underline{V})(t)]^{-1} = 1.$$ (54)

If $s_{\overline{n}|}^{-1} < \delta$, then $(aU)(t)$ increases indefinitely as $t \to \infty$. Again there is a relative kind of convergence. Rearrange formula (50) as follows:

$(aV)(t) - (aP)(t) = [(aV)(0) - (aP)(0)] \exp[-(\ddot{a}_{\overline{n}|}^{-1} - \delta)t]$.

We then divide by $(aV)(t) = (aV)(0)e^{\tau t}$ to obtain

$$\frac{(aP)(t)}{(aV)(t)} = \left[1 - \frac{(aP)(0)}{(aV)(0)}\right] \cdot \exp[-\ddot{a}_{\overline{n}|}^{-1} - (\delta - \tau)]t$$

$$= \left[1 - \frac{(aP)(0)}{(aV)(0)}\right] \exp[-s_{\overline{n}|}^{-1} t] ,$$

where $s_{\overline{n}|}$ is valued at $\delta - \tau$ and n is a finite number. In this case formula (54) holds once more and the same sort of relative convergence takes place. The unfunded supplemental present value gets indefinitely large but the funding ratio $(aP)(t)/(aV)(t)$ approaches 1.

The ratio of the annual contribution rate defined by formula (48) to the rate of payroll payment (W(t)), each evaluated at time t, is a decreasing function of t. To establish this fact we use the same ideas as in the previous subsection's equation (41). We start with equation (48) and substitute equation (50) to obtain

$$\frac{(aC)(t)}{(W)(t)} = \frac{P(t)}{W(t)} + \frac{(aU)(t)}{\ddot{a}_{\overline{n}|} W(t)}$$

$$= \frac{P(0)}{W(0)} + \left[\frac{(aU)(0)}{\ddot{a}_{\overline{n}|} W(0)}\right] \exp[-\ddot{a}_{\overline{n}|}^{-1} t] .$$ (55)

Since $0 < s_{\overline{n}|}^{-1}$, $(aC)(t)/W(t)$ decreases as t increases and

$$\lim_{t\to\infty} \frac{(aC)(t)}{W(t)} = \frac{P(0)}{W(0)} ,$$

that is, the ratio of the contribution rate to the payroll rate tends to the ratio of the normal cost rate to the payroll rate at time zero.

Aggregate Cost Method

It is natural to ask what is the result if for the normal cost plus amortization over a moving term method, discussed in the preceding subsection, the term n is taken equal to $N = \bar{x} - a$, the future funding term for the actuarial cost method defined by the accrual function M(x)? We then have

$(aC)(t) = P(t) + (aU)(t)/\ddot{a}_{\overline{N}|}$ (56)

which may be rearranged as

$(aC)(t) = [P(t)\ddot{a}_{\overline{N}|} + (aV)(t) - (aP)(t)]/\ddot{a}_{\overline{N}|}$

$= [(aA)(t) - (aP)(t)]/\ddot{a}_{\overline{N}|}$, (57)

since $P(t)\ddot{a}_{\overline{N}|} = (Pa)(t)$ by formula (24), and $(aV)(t) - (aA)(t) - (Pa)(t)$. In the notation of formula (56) of [1], $(Pa)(t)/P(t) = \ddot{a}(t)$, a mean temporary annuity. However, because we are discussing the exponential growth case, $\ddot{a}(t) = \ddot{a}_{\overline{N}|}$ is independent of t [1, formula (85)]. We now recognize that

$(aC)(t) = [(aA)(t) - (aP)(t)]/\ddot{a}(t)$ (58)

is the aggregate cost contribution rate defined by formula (58) of [1]. This demonstrates that (for the model plan and exponential growth case) contributions determined by normal cost plus amortization over a moving term N (the future funding term of the given actuarial cost method) are the same as the contributions under aggregate cost funding with mean temporary annuity value $\ddot{a}(t)$. Further, since the contribution rates are the same in the two cases, the same fund $(aF)(t)$ develops, and the funds convergence in relation to $(aV)(t)$ is the same as indicated in the preceding subsection. Finally, formula (55) shows that $(aC)(t)/W(t) \to P(0)/W(0)$ as $t \to \infty$, which is to be expected under the aggregate cost method.

In a more general case, where exponential growth is not assumed, one can show that the contribution rate defined by

$$(aC)(t) = P(t) + \frac{(aU)(t)}{\bar{a}(t)} \qquad (59)$$

is equivalent to the aggregate cost contribution rate in formula (58). In this case, the amortization annuity value $\bar{a}(t)$ may vary with t.

The discussion here is related to the generalized aggregate cost method first described by Trowbridge [5] and further described in Trowbridge and Farr [6, p. 62] where the role of $\bar{a}_N^{-1}$ is played by the constant k.

V. ANALOGOUS THEORY, ALL MEMBERS

In the preceding two sections a body of theory about pension contributions for the subgroup of active members was developed. A parallel theory exists for the whole group, both actives and retireds. The development of this parallel theory is very similar to that for the active subgroup. Therefore it does not seem necessary to present full details. Instead, some of the modifications of key formulas will be given and an outline of the theory indicated.

The concept and corresponding symbols to be used in this section also appear in early sections of this paper or in [1]. For example $P(t)$, the normal cost rate at time t and $(Pa)(t)$ the present value of future normal costs for active members remains as defined in Section II. Table 1 provides a glossary of the needed symbols.

Within the larger framework of this section, we work with $B(t)$ (1, formula (26)), the annual rate of pension outgo at time t, instead of with $T_P(t)$, the terminal funding annual cost rate. For simplicity, the discussion will be limited to the exponential growth case.

Table 1

Basic Symbols

Whole Group			Active Group	
$A(t)$	[1, formula (28)]	Value of Benefits	$(aA)(t)$	Formula (3)
$C(t)$	[1, page 192]	Contribution Rate	$(aC)(t)$	Formulas (41), (48)
$F(t)$	[1, formula (54)]	Fund	$(aF)(t)$	Formula (42)
$U(t)$	[1, formula (49)]	Unfunded Supplement Present Value	$(aU)(t)$	Explanation formula (41)
$V(t)$	[1, formula (37)]	Supplemental Present Value	$(aV)(t)$	Formula (5)

As a replacement of the income allocation equation (7), one now has

$$P(t) + \delta V(t) = B(t) + \frac{dV(t)}{dt} , \qquad (60)$$

which is formula (43) of [1]. For the exponential growth case, this becomes

$$P(t) + \delta V(t) = B(t) + rV(t) ,$$

which, for $\delta \neq r$, may be rearranged as

$$V(t) = [B(t) - P(t)]/(\delta - r) . \qquad (61)$$

Also, in the exponential growth case, there are the following relations:

$$B(t) = \int_r^\infty e^{r(t+r-x)} s(r)b \,_x^r \beta(x)dx \qquad (62)$$

$$= e^{rt} \,_r s(r)b \, \bar{a}_r^{'\beta}$$

where $\bar{a}_r^{'\beta}$ is valued at force of interest r. Then

$$B(t) = (e^{rt} \,_r s(r)b \, \bar{a}_r^{-\beta})(\bar{a}_r^{-\beta}/\bar{a}_r^{-\beta}) \qquad (63)$$

$$= T_P(t)(\bar{a}_r^{-\beta}/\bar{a}_r^{-\beta}) .$$

Thus if δ (the force of interest) > r (the force of total growth), then $B(t) > T_P(t)$. If $\delta < r$, $B(t) < T_P(t)$ and if $\delta = r$, $B(t) = T_P(t)$. $\qquad (64)$

Further, according to the discussion following equation (12), the annual normal cost rate at time t under initial funding, is given by

$$I_{P(t)} = v^{r-a} T_{P(t)}$$ (65)

where, in accordance with the earlier convention, the discount factor is valued at force $\theta = \delta - \tau$. At first we assume $\delta \neq \tau$. Then on substituting from formula (64),

$$I_{P(t)} = B(t)(\bar{a}_r^\beta {}'/\bar{a}_r^\beta)v^{r-a}.$$ (66)

Before proceeding with the development, we will examine the function $(\bar{a}_r^\beta {}'/\bar{a}_r^\beta)$. We have

$$(\bar{a}_r^\beta {}'/\bar{a}_r^\beta) = \int_r^\infty e^{-(\delta-\tau)y}(e^{-\tau(y-r)}{}_{y-r}P_r\beta(y)[\int_r^\infty e^{-\tau(y-r)}{}_{y-r}P_r\beta(y)dy]^{-1}\}dy$$

$$= E[e^{-(\delta-\tau)y}] = e^{-(\delta-\tau)\hat{y}},$$

where the expectation is taken with respect to the density function within the braces. The number $\hat{y}$ falls above r as a result of the mean value theorem for integrals. If $\tau \to \delta$ we find

$$\lim_{\tau \to \delta} \hat{y} = r + E[y - r] = \bar{y},$$

where $\bar{y}$ is the expected value associated with the density function within braces when $\delta = \tau$. Thus $\bar{y}$ may be interpreted as the average age of pension payment.

With this result, we may write equation (66) as

$$I_{P(t)} = B(t)e^{-\theta(\hat{y}-a)}.$$ (67)

If $\delta > \tau$, $I_{P(t)} < B(t)$, if $\delta = \tau$, $I_{P(t)} = B(t)$ and if $\delta < \tau$, $I_{P(t)} > B(t)$.

Next, since $\hat{A}(t) = I_{V(t)}$, the supplemental present value under initial funding, and observing that the expanded income allocation equation, and equation (62)

holds for initial and terminal as well as continuous cost methods, one finds

$$\hat{A}(t) = I_{V(t)} = [B(t) - I_{P(t)}]/(\delta - \tau).$$

By substituting from equation (67), we have

$$\hat{A}(t) = B(t)[1 - v^{\hat{y}-a}]/(\delta - \tau)$$
$$= B(t)\bar{a}_{\overline{\hat{y}-a}}.$$ (68)

Here $\hat{y}$ is determined as in equation (67). From formulas (65) and (67) one has

$$T_{P(t)} = B(t)e^{-\theta(\hat{y}-r)}.$$ (69)

Formula (69) then permits the transformation of formulas (37) or (14) into

$$P(t) = B(t)e^{-\theta(\hat{y}-\hat{x})},$$ (70)

and use of formula (62) leads to

$$V(t) = B(t)\bar{a}_{\overline{\hat{y}-\hat{x}}}.$$ (71)

Finally from formula (71) and (68) we have

$$(Pa)(t) = \hat{A}(t) - V(t) = B(t)v^{\hat{y}-\hat{x}}\bar{a}_{\overline{\hat{x}-a}} = P(t)\bar{a}_{\overline{\hat{x}-a}}$$ (72)

which is a restatement of formula (24).

The following table of correspondences between the formulas of this section and Section III prevails.

Table 2

Correspondence Among Formulas

Entire Group	Active Group (δ > τ)
(Section V)	(Section III)
(68)	(10)
(70)	(15)
(71)	(21)
(72)	(24)

The concepts of this section may be clarified by the Lexis type diagram in Figure 2. The general format is like Figure 1.

Figure 2

Illustration of Relations Among Funding Functions, Whole Population, Exponential Growth
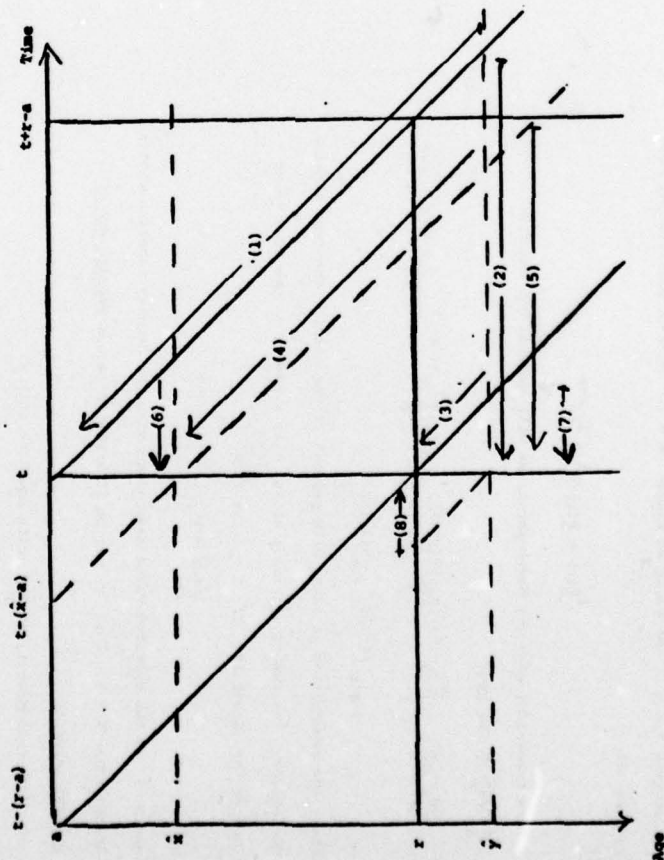
Explanation: The dashed diagonal line may be viewed as the mean path followed by the group active at time t. The paths to be followed in tracing the relationships are indicated by relationship numbers. Payments along horizontal lines are of different amounts because of τ and have different present values because of δ.

Relationships:

(1) Equation (67), $I_P(t) = B(t)e^{-\delta(\bar{y}-a)}$

(2) Equation (68), $A(t) = B(t)\bar{a}_{\overline{y-a}}$

(3) Equation (69), $T_P(t) = B(t)e^{-\delta(\bar{y}-r)}$

(4) Equation (70), $P(t) = B(t)e^{-\delta(\bar{y}-\bar{x})}$

(5) Equation (71), $V(t) = B(t)\bar{a}_{\overline{y-x}}$

(6) Equation (72), $(Pa)(t) = P(t)\bar{a}_{\overline{x-a}}$

Additional relationships can be seen by breaking $A(t)$ and $V(t)$ into components for active and retired lives. For example, $(rV)(t)$, the present value of benefits for retired lives is represented by $B(t)\bar{a}_{\overline{y-r}}$, relationship (7), or by $T_P(t)\bar{s}_{\overline{y-r}}$, relationship (8).

Of course, one may go on to examine the formulas that correspond to (31) to (34) for δ → τ and to (36) to (39) for δ < τ. The Theory of Section IV concerning contributions for active lives may be recast in terms of the whole group. The changes are primarily notational, except that the total rate of pension outgo $B(t)$ plays the role of the terminal funding cost $T_P(t)$ and the global income allocation formula (61) is used.

Proof of Formula (30), $\delta > \tau$

The pension purchase density function associated with entry age normal funding is given by

$$m(x) = \frac{e^{-(\delta-\gamma)x}s(x)\ell_x}{\int_a^r e^{-(\delta-\gamma)y}s(y)\ell_y dy}, \qquad 0 \le x < r .$$

Upon substitution in formula (13), and using $\theta = \delta - \tau = \delta - a - \gamma > 0$, we have

$$e^{\theta\hat{x}} = \int_a^r e^{\theta x}\left\{e^{-(\delta-\gamma)x}s(x)\ell_x \Big/ \int_a^r e^{-(\delta-\gamma)y}s(y)\ell_y dy\right\}dx$$

$$e^{\theta\hat{x}} = \int_a^r e^{-ax}s(x)\ell_x dx \Big/ \int_a^r e^{-(\delta-\gamma)y-ay}s(y)\ell_y dy .$$

so that

$$e^{-\theta\hat{x}} = \int_a^r e^{-(\delta-\tau)y}s(y)\ell_y \Big/ \int_a^r e^{-ax}s(x)\ell_x dx\, dy .$$

The function within the brackets may be interpreted as a density function.

Using Jensen's inequality we have

$$e^{-\theta\hat{x}} = E[e^{-\theta y}] > e^{-\theta E[y]} .$$

Now if $e^{-\theta y}\ell_y$ is a decreasing function $E[y] < \frac{a+r}{2}$. Using this result we may strengthen the previous inequality

$$e^{-\theta\hat{x}} > e^{-\theta E[y]} > e^{-\theta\left(\frac{a+r}{2}\right)}$$

or

$$\hat{x} < \frac{a+r}{2}$$

$$\hat{x} - a = N < \frac{r-a}{2} .$$

VI. CONCLUSION

In this paper a theory of contributions to fund pensions during workers' periods of employment, under dynamic economic and demographic assumptions, has been developed. Relationships among the contribution patterns that may arise under different cost methods have been developed. The theory has economic implications. For example, if the sum of the rate of population increase and salary increases exceeds the interest rate, terminal funding normal cost is below that of any cost method funding pensions during the working lifetimes of members.

REFERENCES

1. Bowers, Newton L., Jr., Hickman, J. C., and Nesbitt, C. J., "Introduction to the Dynamics of Pension Funding". TSA, Vol. 28 (1976), 177.

2. Bowers, Newton L., Jr., "An Approximation of the Distribution of Annuity Costs", TSA, Vol. 19 (1967), 295.

3. Cooper, Steven L., and Hickman, J. C., "A Family of Accrued Benefit Actuarial Cost Methods", TSA, Vol. 19 (1967), 53.

4. Olson, Gary E. "On Some Actuarial Inequalities", TSA, Vol. 27 (1975), 11, discussion by Hans U. Gerber and Donald A. Jones, p. 24.

5. Trowbridge, Charles L., "The Unfunded Present Value of Pension Funding Methods", TSA, Vol. 25 (1963), 15.

6. Trowbridge, C. L., and Farr, C. E., The Theory and Practice of Pension Funding. Homewood, Ill: Richard D. Irwin, Inc. 1976.

7. Trowbridge, J. R., "Assessmentism - An Alternative to Pension Funding", JIA, Vol. 104 (1977), 173.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1864	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE DYNAMICS OF PENSION FUNDING: CONTRIBUTION THEORY		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James C. Hickman, C. J. Nesbitt and N. L. Bowers		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Applica- tions of Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE July 1978
		13. NUMBER OF PAGES 35
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Pension funding
Population growth
Salary growth
Benefit growth
Convergence of funds

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general model for a pension plan involving growth with respect to the population, salaries and retirement benefits is used to study contribution patterns that may arise under different actuarial cost methods. Detailed results are presented for the case where the growth of population and salaries are described by exponential functions. Economic implications are presented and discussed.

DD FORM 1473   EDITION OF 1 NOV 65 IS OBSOLETE